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MANGOES AND BLUEBERRIES

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Dedicated to the memory of Paul Erdős

We prove the following conjecture of Erdős and Hajnal:

For every integer k there is an $f(k)$ such that if for a graph G, every subgraph H of G has a stable set containing $\frac{|V(H)|-k}{2}$ vertices, then G contains a set X of at most $f(k)$ vertices such that $G-X$ is bipartite.

This conjecture was related to me by Paul Erdős at a conference held in Annecy during July of 1996. I regret not being able to share the answer with him.

Introduction

A graph G is k-*near bipartite* if every set of t vertices of G contains a stable set of size $\frac{t-k}{2}$. Clearly, G is bipartite if and only if it is 0-near bipartite and if $G-X$ is bipartite then G is $|X|$-near bipartite. Erdős and Hajnal (see [4]) conjectured that a partial converse holds, as follows:

The Conjecture. *For every k there is an $f(k)$ such that if a graph G is k-near bipartite then there is some set X of at most $f(k)$ vertices such that $G-X$ is bipartite.*

We prove this conjecture. In doing so, we find it convenient to use the term *odd cycle cover* to denote a set of vertices X such that $G-X$ is bipartite.

Remark. The original definition of near-bipartite graphs was different. The desired stable set need only exist for graphs with an even number of vertices and the $-\frac{k}{2}$ was replaced by $-k$ in the bound on the stable sets size. The present definition seems more natural (e.g. the bipartite graphs are precisely the 0-near bipartite graphs) and the conjecture holds for one definition if and only if it holds for the other.

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Note that if a graph contains $k+1$ vertex disjoint odd cycles then it is not k-near bipartite. So if the Erdős–Pósa property held for odd cycles, i.e. if the minimum number of vertices in an odd cycle cover were bounded by a function of the size of a maximum packing of vertex disjoint odd cycles, then the conjecture would follow immediately (the term Erdős–Pósa originated because of [3] in which it is shown that the family of cycles has this property). However, as pointed out by Lovász and Schrijver [5], the Erdős–Pósa property does not hold for odd cycles, as we now show.

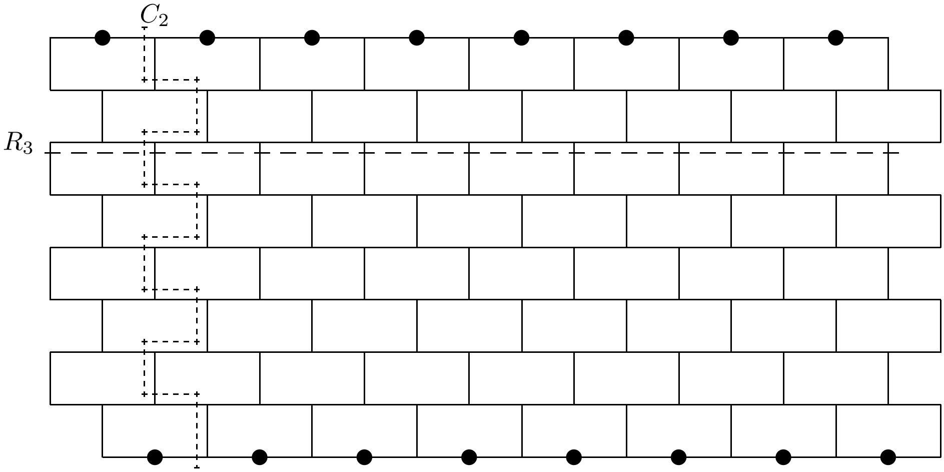


Figure 1. An elementary wall of height 8

An elementary wall of height eight is depicted in Figure 1. An *elementary wall* of height h for $h \geq 3$ is similar. It consists of h levels each containing h bricks, where a brick is a cycle of length six. A *wall* of height h is obtained from an elementary wall of height h by subdividing some of the edges, i.e. replacing the edges with internally vertex disjoint paths with the same endpoints(see Fig. 2).

An *Escher wall of height h* consists of a wall W of height h together with h vertex disjoint paths $P_1,\ldots,P_h$ such that:

(i) Each P_i has both endpoints on W but is otherwise disjoint from W.

(ii) One endpoint of P_i is in the ith brick of the top level of bricks of W, the other is in the $(h+1-i)$th brick of the bottom row of W. Furthermore, both of these vertices are in only one brick of W.

(iii) W is bipartite but for each i, $W \cup P_i$ contains an odd cycle.

See Figure 3 for an example.

Lovász and Scrijver [5] noted that an Escher wall W of height h contains neither 2 vertex disjoint odd cycles nor an odd cycle cover with fewer than h vertices (the first fact follows from the fact that for any Escher wall $W, P_1, \ldots, P_k$, the planar embedding of W can be extended to an embedding of the Escher wall in

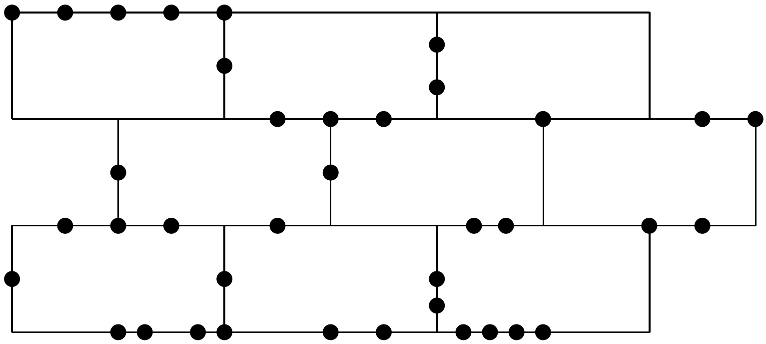


Figure 2. A wall of height 3

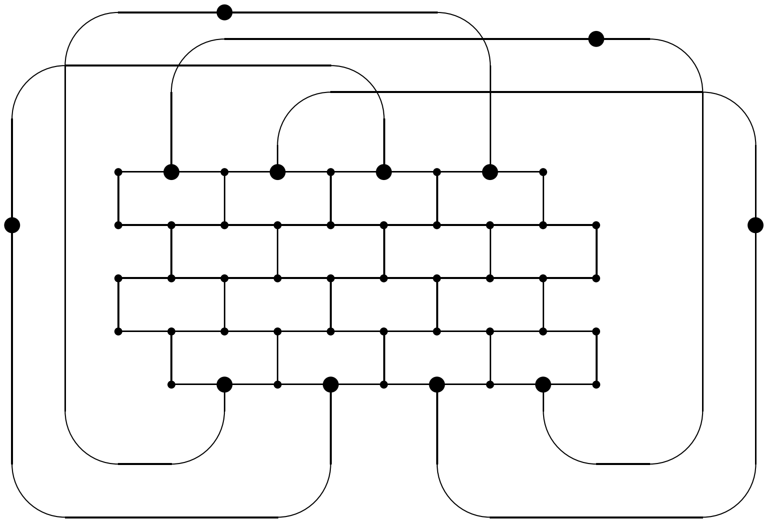


Figure 3. An Escher wall of height 4

the projective plane so that every odd cycle is non-null homotopic by routing the P_i through a cross-cap; the fact that there is no odd cycle cover of size $h-1$ follows from the fact that if X is a set of $h-1$ vertices then X fails to intersect some path along the top of a level of bricks of W and similarly there is some i such that P_i is disjoint from X and both endpoints of P_i are connected to this row in $W-X$. Hence the two endpoints of this P_i are connected by a path Q in $W-X$ and then P_i+Q is an odd cycle in $G-X$). This shows that the Erdős–Pósa property does not hold for odd cycles (in fact, it holds for the cycles of length p *mod* m if and only if p is congruent to 0 *mod* m see [15] and [1]).

However, we are able to show that in a certain sense, Escher walls are the only obstruction preventing the Erdős–Pósa property from holding. To wit, we show:

Theorem 1. *For all k and w there is a $t(k,w)$ such that if G is a graph with neither k vertex disjoint odd cycles nor an Escher wall of height w then there is an odd cycle cover X of G with $|X| \le t(k,w)$.*

With the help of the following technical result, this yields a proof of the conjecture:

Theorem 2. *For all k there is a $g(k)$ such that any Escher wall of height at least $g(k)$ is not k-near bipartite.*

Corollary. *For all k there is an $f(k)$ such that if G is k-near bipartite then G has an odd cycle cover of order at most $f(k)$.*

Proof. Set $f(k) = t(k+1, g(k))$. Since no k-near bipartite graph has $k+1$ vertex disjoint odd cycles, the result follows. ∎

Now, the following result is an easy corollary of Theorem 2.

Definition. A *half integral packing* of k odd cycles is a set of $2k$ odd cycles such that each vertex is in at most two of these odd cycles.

Theorem 3. *For all k there is an n_k such that a graph without a half integral packing of k odd cycles has an odd cycle cover of order at most n_k.*

Proof. To prove this theorem, it is sufficent to prove that every Escher wall of height $2h$ contains a half-integral packing of h odd cycles. The proof is suggested by the diagram in Figure 4. We delay the straightforward proof to Section 7 so as to defer some technical definitions. ∎

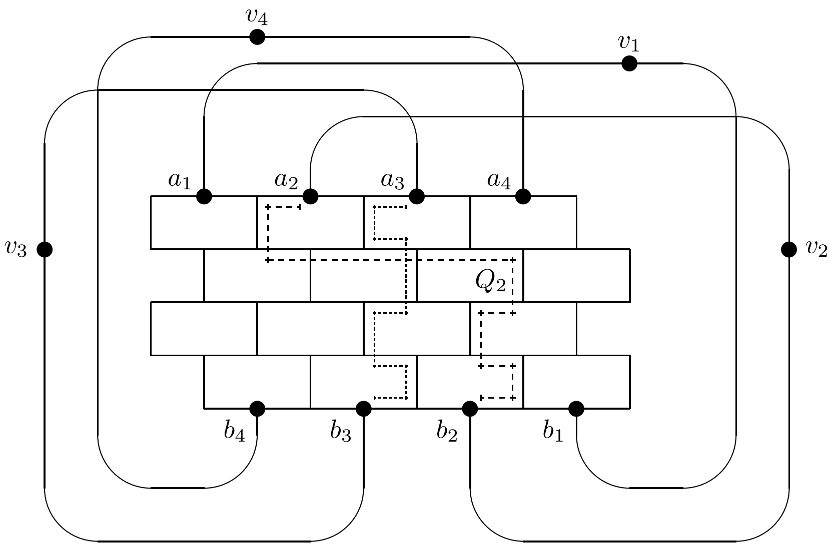


Figure 4

The proof of Theorem 2 is quite straightforward. Theorem 1 is more difficult. We attack it using the powerful graph minor theory developed by Robertson and Seymour. Our approach is to split each vertex of a minimal odd cycle cover into two, thereby producing a bipartite auxiliary graph and then to analyze the set of paths in the auxiliary graph corresponding to odd cycles in the original graph.

Specifically, we define for every graph G and odd cycle cover $C = \{c_1,\dots,c_l\}$ of G, an auxiliary bipartite graph $G' = G'(G,C)$. To do so, we first choose an arbitrary partition of $G-C$ into two stable sets A' and B'. We set $V(G') = V(G) - C \cup \{c^A, c^B | c \in C\}$ and $E(G') = E(G-C) \cup \{c^A y | c \in C, y \in B', cy \in E(G)\} \cup \{c^B y | c \in C, y \in A', cy \in E(G)\} \cup \{c_i^A c_j^B | c_i, c_j \in C, c_i c_j \in E(G), i < j\}$. We note that G' is bipartite with bipartition $(A = A' \cup \{c^A | c \in C\}, B = B' \cup \{c^B | c \in C\})$. For any vertex c in C we define *image(c)* as $\{c^A, c^B\}$ and $preimage(c^A) = preimage(c^B) = c$. For any vertex x in $G-C$, we set $image(x) = preimage(x) = x$. For any set S of vertices of G, the *image of* S is the union of the images of its elements. Similarly, for any set S of vertices of G', the *preimage of* S is the union of the preimages of its elements. Now, each edge e of G' corresponds to an edge of G whose endpoints are the preimages of the endpoints of e. We call this edge the *preimage* of e. The preimage function extends to subgraphs in a natural manner.

Now, for any subset $X = \{x_1,\dots,x_k\}$ of C, if there are k vertex disjoint paths $P_1,\dots,P_k$ in $G' - (image(C) - image(X))$, such that P_i links x_i^A with x_i^B then the preimages of these paths form k vertex disjoint odd cycles of G. It is this observation which allows us to resolve the conjecture using graph minor theory, for this theory has a lot to say about graphs in which the desired paths do not exist. The only difficulty is that we forbid a set of k such paths which are internally disjoint from $image(C)$, whilst the techniques of Robertson and Seymour apply only if this condition is dropped.

If we drop this condition, then instead of corresponding to a set of vertex disjoint odd cycles, the paths we obtain would correspond to a set of odd cycles, which use each vertex at most twice. In fact, our initial analysis only allows us to insist that a vertex is used at most four times in any odd cycle. Thus, we obtain:

Theorem 4. *For all i there is a $p(i)$ such that if G is a graph which does not contain a set of i odd cycles such that each vertex is in at most four of these cycles then there is a set X of at most $p(i)$ vertices s.t. $G-X$ is bipartite.*

We then prove:

Theorem 5. *For all s there is a $q(s)$ such that if G is a graph containing a set of $q(s)$ odd cycles, such that no vertex is in more than four of these cycles then G contains either an Escher wall of height s or s vertex disjoint odd cycles.*

Combining these results implies Theorem 1 with $t(k,w) = p(q(max(k,w)))$. It remains only to prove Theorems 2, 4, and 5.

To prove Theorem 5, we will again consider a minimal odd cycle cover, corresponding auxiliary graph, and sets of disjoint paths. However, the auxiliary graph we construct is slightly different. We take advantage of the fact that a minimal counterexample to this theorem consists of the union of a set of vertex disjoint odd cycles hitting each vertex at most four times and hence has maximal degree at most eight. So, by splitting the vertices of our minimal cover into up to eight vertices instead of two, we can ensure that at most one of the vertices in the image of an element of C has degree more than one. Now, we no longer need to worry about a vertex in C being the interior vertex of two paths. Except for this slightly different definition of the auxiliary graph, the proofs of Theorems 4 and 5 are essentially identical. Details follow.

2. Tree width and walls

Our ultimate goal is to prove that the Erdős–Pósa property holds for the odd cycles in graphs in which the height of Escher walls is bounded. In this section we prove the analagous result for graphs in which the height of all walls is bounded. To do so, we present necessary and sufficent conditions for a graph not to have high walls. The machinery we introduce will be extremely useful in the proof of Theorem 4. Actually, it is extremely useful in many contexts. For a more comprehensive treatment of this topic, readers may consult [6].

A *tree decomposition* $[T,\mathcal{X}]$ of a graph G consists of a tree T and a set $\mathcal{X} = \{X_t | t \in T\}$ of subsets of $V(G)$ such that

(i) the set $S_v = \{t | v \in X_t\}$ induces a non-empty subtree of T for every $v \in V(G)$, and

(ii) for every edge uv of G there is a node $t \in T$ such that $\{u,v\} \subseteq X_t$.

For any subtree S of T we define G_S to be the subgraph of G induced by $\cup_{t\in S} X_t$. We note that if st is an arc of T and S and R are the components of $T - st$ with $s \in S$ then $X_s \cap X_t$ is a cutset separating $G_S - (X_s \cap X_t)$ from $G_R - (X_s \cap X_t)$. The *width* of a tree decomposition $[T,\mathcal{X}]$ is $max(|X_t|, t \in T) - 1$. The *tree width* of G is the minimum of the widths of its tree decompositions.

A *strong preference* B (or simply *s-preference*) of order l in G is a function mapping each subset X of fewer than l vertices of G to a component of $G - X$ such that for any three such sets X_1, X_2, X_3 we have either:

(i) $B(X_1) \cap B(X_2) \cap B(X_3) \neq \emptyset$, or

(ii) some edge of G has an endpoint in each of $B(X_1), B(X_2)$, and $B(X_3)$.

It is not that difficult to see that if a graph has a strong preference of order l then its tree width must be at least $l-1$. Robertson and Seymour proved a partial converse showing:

Theorem 2.1. *If G has tree width at least l then it has a strong preference of order at least $\frac{2}{3}l$.*

Remark. *Precise duality results are given in ([9], Theorem 5.2]) where strong preferences are called tangles, and [12]. See [6] which shows the equivalence between tangles and strong preferences.*

Now, every wall of order h yields a s-preference of order $h-1$, we simply let $B(X)$ be the component of $G-X$ which contains at least one path from the top to the bottom of W, and hence also a path from the left of W to the right. We omit the straightforward proof that this is a s-preference of order h–1. The reader should be able to convince herself that it is a s-preference of order $\frac{h-1}{3}$. We actually need to apply a dual result of Robertson, Seymour, and Thomas (this is an immediate corollary of [11], Theorem 2.3; see [7] for a precursor) which is much more difficult:

Theorem 2.2. *For every even integer $h \geq 6$, if G contains a s-preference B of order 20^{64h^5} then G contains a wall W of height h. Furthermore, we can find such a wall W which is controlled by B, i.e. such that for every subset X of at most $h-1$ vertices, $B(X)$ is the unique component of $G-X$ containing a path from the top row of W to the bottom row of W.*

We close this section by proving that the Erdős–Pósa property does indeed hold for the odd cycles in graphs of bounded tree width. The precise result we obtain is a weakening of a result of Thomassen [15]:

Theorem 2.3. *If G is a graph of tree width at most ω containing no $r+1$ vertex-disjoint odd cycles, then there is an odd cycle cover of G containing at most $2r{*}(\omega{+}1)$ vertices.*

Proof. We prove the theorem by induction on r. If $r=0$ then there is nothing to prove. If $r=1$ then consider a tree decomposition $[T,\mathcal{X}]$ for G of width at most ω. If any X_t intersects every odd cycle we are done. Otherwise, for each t in T we can find a neighbour s of t such that for the subtree S of $T-t$ containing s, G_S-X_t contains an odd cycle. We orient the edge ts to s. Since T has more nodes than arcs, there is an arc st oriented in both directions. Letting S be the subtree of $T-st$ containing s, and R the component of $T-st$ containing t, we see that the two disjoint subgraphs G_S-X_t and G_R-X_s of $G-(X_s \cap X_t)$ both contain an odd cycle, contradicting $r=1$.

Now suppose $r \geq 2$ and the theorem holds for all smaller values of r. Let $[T,\mathcal{X}]$ be a tree decomposition of G of width at most ω. As above, we can find either an odd cycle cover of size at most $\omega+1$ for G or an arc st of T such that at least two components of $G'=G-(X_s \cap X_t)$ contain odd cycles. Now, applying induction to the components of G' separately, we see that there is an odd cycle cover X' of G' containing at most $(\omega+1)*(2r-2)$ vertices. Now, $X' \cup (X_s \cap X_t)$ is an odd cycle cover of G. ∎

3. Cutset structure and a useful preference

We are now ready to begin the proof of Theorem 4. We define the values of the $p(i)$ inductively. That is, given that $p(0),\ldots,p(i-1)$ exist, we choose $p(i)$ to be some integer which satisfies certain inequalities given below in terms of $i, p(0),\ldots,p(i-1)$. Rather than specifying $p(i)$ explicitly, we choose the minimum integer satisfying all the inequalities scattered throughout the next three sections. The first of these is that $p(i)$ is at least $2p(i-1)+1$.

Our proof also proceeds inductively. The theorem holds for $i=0$ with $p(0)=0$, since every graph contains a set of 0 vertex disjoint odd cycles. It also holds for $i=1$ with $p(1)=0$. So, we assume that we have proven the theorem for $i<r$ and prove it for $i=r$.

To begin, we assume that Theorem 4 fails to hold for r and choose a counterexample G, a minimal odd cycle cover C for G, and corresponding auxiliary graph G'. Given a set X in G', we say that $x \in C$ *crosses* X if there is no component of $G'-X$ containing both x^A and x^B. We let $Crosses(X)$ be the set of vertices of C which cross X. A *separation* (W,Y) of G consists of two edge disjoint subgraphs whose union is G. For a set $D \subseteq V(G')$ and a component K of $G'-D$, we often use $(D \cup K, G'-K)$ to denote the separation (W,Y) where W is the graph induced by $V(W)=V(K)\cup D$, and $V(Y)=V(G')-V(K), E(Y)=E(G')-E(W)$. Given a separation (W,Y) of G' we say x in C *crosses* (W,Y) if its image intersects both $V(W)$ and $V(Y)$. we let $Cross(W,Y)$ be the set of vertices which cross (W,Y).

3.1. *Let X be any set of vertices in G' and (W,Y) any separation of G' with $V(W)\cap V(Y)=X$. Then $Cross(W,Y)$ contains at most $|X|$ vertices.*

Proof. We can partition $G'-X$ into the two stable sets, $(A\cap W-X)\cup(B\cap Y-X)$ and $(B\cap W-X)\cup(A\cap Y-X)$. Now, $G-(C-Cross(W,Y))-preimage(X)$ is obtained by repeatedly identifying vertices on the same side of this partition and hence is bipartite. Thus, $(C-Cross(W,Y))\cup preimage(X)$ is an odd cycle cover of G. (3.1) follows by the minimality of C. ∎

3.2. *For any set X of vertices of G', the set $Crosses(X)$ has at most $(|X|+1)|X|$ elements.*

Proof. By (3.1), no component U of $G'-X$ contains more than $|X|$ elements of $image(Crosses(X))$; consider the separation $(U\cup X, G'-U)$ Choose a separation (W,Y) of G', with $W\cap Y=X$, such that there is no element x of $Crosses(X)$ for which both x^A and x^B are in W(i.e. s.t. $Cross(W,Y) = Crosses(X)\cap preimage(W)$; such a choice is possible because for any component U of $G'-X$, $(U\cup X, G'-U)$ is a possible choice), and subject to this with W maximal. By (3.1), $|Cross(W,Y)| \le |X|$. If there is a component U of $Y-X$ containing no element of $image(Cross(W,Y))$ then $(W\cup U, Y-U)$ contradicts the maximality of W. It follows that $Y-X$ has at most $|Cross(W,Y)|$ components. Hence by (3.1), $|Crosses(X) \cap preimage(Y-X)| \leq |Cross(W,Y)||X| \leq |X|^2$. Since $|Crosses(X) \cap preimage(X)| \leq |X|$, the result follows. ∎

3.3. *For any cutset X with fewer than $\sqrt{p(r)-2p(r-1)}-1$ vertices, there is no component U of $G'-X$ such that for some x_1, x_2 in C, the image of x_1 is contained in $V(U)$ and there is an x_2^A to x_2^B path in $G'-U$.*

Proof. Otherwise, since $preimage(U)$ contains an odd cycle through x_1, there is an odd cycle cover of $preimage(G'-U) - Crosses(X)$ with at most $p(r-1)$ vertices. Similarily, there is an odd cycle cover of $preimage(U) - Crosses(X)$ with at most $p(r-1)$ vertices. The union of these two odd cycle covers with $preimage(X) \cup Crosses(X)$ is an odd cycle cover of G. By (3.2) and our condition on the size of X, the result follows. ∎

3.4. *For any set X of fewer than $\sqrt{p(r)-2p(r-1)}-1$ vertices of G' there is a component U of $G'-X$ which contains the image of $C - Crosses(X)$, and hence contains the image of all but at most $(|X|+1)^2$ of the elements of C.*

Proof. The first statement is an immediate corollary of (3.3), the second then follows from (3.2). ∎

Theorem 3.5. *There is a s-preference B assigning to each subset X of at most $j = \frac{\sqrt{p(r)-2p(r-1)}}{2} - 1$ vertices of G' the component of $G'-X$ which contains the image of all but at most $(|X|+1)^2$ of the elements of C.*

Proof. By (3.4) this function is well defined. Furthermore, any three such connected subgraphs must intersect in an element of $image(C)$ so the function is a s-preference. ∎

For each set X of at most j vertices, we shall call $B(X)$ *the big component of X*.

4. Routing through clique minors

The main goal of Section 3 was to show that our minimal counter-example to Theorem 4 must have a reasonably complicated structure, to wit the odd cycle cover defines a preference in the auxiliary graph. In this section, we show that on the other hand, our minimal counterexample to Theorem 4 cannot have too complicated a structure, to wit that the auxiliary graph cannot have a large clique minor controlled by the preference so defined. We begin with the appropriate definitions and a result of Robertson and Seymour.

For our purposes, a *clique minor M of order m* in H consists of a set $\{C_1, \ldots, C_m\}$ of vertex disjoint connected subgraphs between every two of which there is an edge. Note that for any set Z of at most $m-1$ vertices of H there is a unique component of $H-Z$ which contains some (or equivalently all) C_i disjoint from X. We say a set Z separates a set X from the clique minor if the component of $H-Z$ completely containing some C_i is disjoint from X. A s-preference $\mathcal{B}$ *controls* M if $\mathcal{B}$ has order at least m and for every set X of fewer than m vertices $\mathcal{B}(X)$ completely contains some C_i.

Robertson and Seymour([10], Theorem 5.4) proved:

4.1. *Let X be a set of at most $2k$ vertices in a graph H and let $M=\{C_1,\dots,C_m\}$ be a clique minor of order at least $8k+1$ in H such that there is no Z in H with $|Z|<|X|$ separating X from M. Then for any set $\{(s_1,t_1),\dots,(s_l,t_l)\}$ of disjoint pairs of vertices of X there are l vertex disjoint paths $P_1,\dots,P_l$, internally vertex disjoint from X such that P_i has endpoints s_i and t_i.*

Remark. *If each C_i is a vertex then this theorem is easily seen to be true, for then the clique minor is actually a clique and by Menger's theorem there are $|X|$ vertex disjoint paths $Q_1,\dots,Q_{|X|}$ from X to the clique. Now, for any set of pairs of endpoints, the desired P_i can be found by using the Q_j and the appropriate edges of the clique.*

Using (4.1) we can prove the main result of this section,

4.2. *G' contains no clique minor of order $8r+1$, controlled by the s-preference B of Theorem 3.5.*

Proof. Assume the contrary, and let $M=\{C_1,\dots,C_q\}$ be a clique minor of order $8r+1$ controlled by B. Choose a maximum size subset $X=\{x_1,\dots,x_l\}$ of C such that there is no set of fewer than $2|X|$ vertices separating $image(X)$ from M. Assume first that $l=0$. In this case, let x be any vertex of C. Since $l=0$, we know that there is a vertex y such that y separates $image(x)$ from M and hence $image(x)$ is disjoint from the big component of $G'-y$. Let P be a path between x^A and x^B in G'. Clearly although P may pass through y, it is disjoint from the big component of $G'-y$. However, this contradicts (3.3). So, we can assume that X is non-empty.

We know that there do not exist r vertex disjoint paths in G' each linking some x_i^A to x_i^B, as this would yield a half-integral packing of r odd cycles. So, by (4.1), we can assume $l<r$. We do so and derive a contradiction. Let Z_0 be a set of $2l$ vertices separating $image(X)$ from M chosen so that the big component U_0 of $G'-Z_0$ which completely contains some C_i is as small as possible (note that $Z_0=image(X)$ may occur).

We know that for all but $(2r)^2$ of the x in C, $image(x)$ is contained in U_0. For each such x, we know by the maximality of X, that there is a set of at most $2l+1$ vertices which separates $image(X+x)$ from M. For each such x, we choose such a set Z_x so as to minimize the size of the big component U_x of $G'-Z_x$ (which completely contains some C_i).

We claim that for each x with $image(x)\subseteq U_0$, Z_x is contained in $U_0\cup Z_0$ and also separates Z_0 from M. To see this, let U_x' be the big component of $G'-Z_x-Z_0$

(which completely contains some C_i). Let A_1 be the set of vertices of $Z_0 \cup Z_x$ incident to a vertex of U'_x. Let A_2 be the set of vertices of $Z_0 \cup Z_x$ in $image(X)$ or incident to a component of $G' - Z_0 - Z_x$ containing a vertex of $image(X)$. Note that a vertex which is in both A_1 and A_2 is also in both Z_0 and Z_x. Thus, $|A_1| + |A_2| \leq |Z_0| + |Z_x|$. Now, our choice of X ensures that $|A_2| \geq 2l$. Thus, $|A_1| \leq |Z_x|$ and so, since $U'_x \subseteq U_x$, our choice of Z_x ensures that $Z_x = A_1$. Since A_1 separates Z_0 from M by definition, our claim follows.

We also claim that there is no x with $image(x) \subseteq U_0$ such that $Z_0 \subseteq Z_x$. For, assume the contrary. Then $Z_x - Z_0$ is a vertex y. Now, there is a path between x^A and x^B in U_0 and obviously any such path is disjoint from U_x. But this contradicts (3.3) because U_x is the big component of $G' - Z_x$.

If we insist that $p(r)$ is at least $(2^{2r})(4r^2+4r+2)+4r^2$ then we can find a set of $4r^2+4r+2$ vertices of C, $X' = \{x'_1, \ldots, x'_{4r^2+4r+2}\}$ such that for $1 \leq i \leq 4r^2+4r+2$ we have $image(x'_i) \subseteq U_0$ and $Z_{x'_i} \cap Z_0 = Z_{x'_1} \cap Z_0$.

We claim that we can assume there is a j such that $Z_{x'_j} \neq Z_{x'_1}$. Otherwise, consider the big component U_1 of $G' - Z_{x'_1}$. By (3.3), every vertex of X' is in $Crosses(Z_{x'_1})$. Hence, the $4r^2+4r+2$ vertices in $X \cup X'$ contradict (3.2) for $Z_{x'_1}$. This proves our claim, we can assume $j=2$. We let $Z_1 = Z_{x'_1}$ and $Z_2 = Z_{x'_2}$.

Now, let U_3 be the big component of $G' - Z_1 - Z_2$. Let Z_3 be the set of vertices of $Z_1 \cup Z_2$ incident to U_3. Let Z_4 be the set of vertices of $Z_1 \cup Z_2$ which are either in Z_0 or incident to a vertex in a component of $G' - Z_1 - Z_2$ containing a vertex of Z_0. Note that if a vertex is in both Z_3 and Z_4 then it is in both Z_1 and Z_2. Thus $|Z_1| + |Z_2| \geq |Z_3| + |Z_4|$. Now, U_3 is contained in $U_{x'_1}$ and $U_{x'_2}$ and one of these containments is strict because $Z_1 \neq Z_2$. Thus, by the choice of Z_1 and Z_2, we have $|Z_3| \geq 2l+2$. So $|Z_4| = 2l$, and by our choice of Z_0, $Z_4 = Z_0$. Recall that $Z_2 \cap Z_0 = Z_1 \cap Z_0$, so we have obtained: $Z_0 \subseteq Z_1$. However, we have already shown this to be impossible, so our assumption fails and (4.2) holds. ∎

5. Graphs without clique minors

In Section 3 we proved that a minimal counterexample to Theorem 4 cannot have bounded tree width, i.e. it must contain a large order s-preference. In Section 4, we proved that this s-preference cannot control a large clique minor. In this section, we complete the proof of Theorem 4 by applying a structural result of Robertson and Seymour concerning graphs which have a large s-preference controlling no large clique minor. To state this result we will need a few definitions.

The *nails* of a wall are the vertices of degree three within it and its corners. Any wall has an essentially unique planar embedding. We define a distance function d_W on the vertices of W so that $d_W(x,y)$ is the minimum number of regions of this embedding that an arc in the plane with endpoints x and y intersects. The *perimeter* of a wall W, denoted $per(W)$ is the unique cycle bounding a face in this embedding which contains more than 6 nails. We also define an *internal distance function* $intd_W$ on the vertices of W so that $intd_W(x,y)$ is the minimum number of regions of this embedding that an arc in the plane with endpoints x and y *which does not intersect the face bounded by the perimeter* intersects. For any wall W in a graph H, there is a unique component U of $H-per(W)$ containing $W-per(W)$. The *compass* of W, denoted $comp(W)$, consists of the graph with vertex set $V(U)\cup V(per(W))$ and edge set $E(U)\cup E(per(W))\cup\{xy|x\in V(U),y\in V(per(W))\}$ (note that this may exclude edges between vertices of $per(W)$). A *subwall* of a wall W is a wall which is a subgraph of W. A subwall of W of height h is *proper* if it consists of h consecutive bricks from each of h consecutive rows of W. The *exterior* of a subwall W' is $W-W'$. A proper subwall is *dividing* if its compass is disjoint from its exterior. We say a proper subwall W' of W is *dividing in a subgraph* H of F if $W'\subseteq H$ and the compass of W' in H is disjoint from $(W-W')\cap H$. Note that:

5.1. *If two proper subwalls of a wall W which are dividing in H are disjoint then so are their compasses in H.*

We note that an elementary wall of height k can be decomposed into $k+1$ disjoint horizontal paths, which we enumerate, from top to bottom, as $R_1,\dots,R_{k+1}$. It also contains $k+1$ columns where each column contains $2k-1$ edges, one from each row except the first and the last and $k+1$ vertical edges (we omit the fussy details, see Figure 1). We enumerate the columns from left to right as $C_1,\dots,C_{k+1}$. The columns and rows of arbitrary walls are defined similarly. The *corners* of a wall are $R_1\cap C_1$, $R_1\cap C_{k+1}$, $R_{k+1}\cap C_1$, $R_{k+1}\cap C_{k+1}$. The *pegs* of a proper subwall W' of a wall W are the nails of W on the perimeter of W' which are not nails of W'. A wall is *flat* if its compass does not contain two vertex disjoint paths connecting the diagonally opposite corners. Note that if the compass of W has a planar embedding whose infinite face is bounded by the perimeter of W then W is clearly flat. Seymour [13], Thomassen [14], and others have characterized precisely which walls are flat. It is easy to see that any subwall of a flat wall must be both flat and dividing. Furthermore, if x and y are two vertices of a flat wall W and there is a path between them which is internally disjoint from W then either x and y are both on $per(W)$ or some brick contains both of them.

Two easy technical results concerning walls are:

(5.2). *Let S be a set of $2m$ vertices on the perimeter of a wall W so that for any pair (x,y) of elements of S, $intd_W(x,y)$ is at least 10 and x is at internal distance at least 10 from every corner of W. Then for any partition of S into m pairs $\{(a_1,b_1),\dots,(a_m,b_m)\}$ there are m paths in W: $P_1,\dots,P_m$ such that*

(i) *a_i and b_i are the endpoints of P_i and appear on no other path,*

and

(ii) *no vertex appears in more than two of the paths.*

Proof. We note that our conditions on S ensure that the height h of W is at least $4m$. We partition S into $S_{top}=S\cap R_1$,$S_{bottom}=S\cap R_{h+1}$, $S_{left}=S\cap C_1$, and

$S_{right} = S \cap C_{h+1}$. For each vertex x of S_{top}, we choose a column C_i such that $i \equiv 0 \pmod 3$, let P_x be the subpath of R_1 from x to $C_i \cap R_1$ and let $A_x = P_x \cup C_i$. Furthermore, we choose C_i closest to x, i.e. so that there is no $j = 3k$ such that $C_j \cap R_1$ is an interior vertex of P_x. Note that this ensures that if x and y are both in S_{top} then A_x and A_y are disjoint. For each vertex x of S_{bottom}, we choose a column C_i such that $i \equiv 1 \pmod 3$, let P_x be the subpath of R_{h+1} from x to $C_i \cap R_{h+1}$ and let $A_x = P_x \cup C_i$. Furthermore, we choose C_i closest to x, i.e. so that there is no $j = 3k+1$ such that $C_j \cap R_{h+1}$ is an interior vertex of P_x. Note that this ensures that if x and y are both in S_{bottom} then A_x and A_y are disjoint. For each vertex x of S_{left}, we choose a row R_i such that $i \equiv 0 \pmod 3$, let P_x be the subpath of C_1 from x to $R_i \cap C_1$ and let $A_x = P_x \cup R_i$. Furthermore, we choose R_i closest to x, i.e. so that there is no $j = 3k$ such that $R_j \cap C_1$ is an interior vertex of P_x. Note that this ensures that if x and y are both in S_{left} then A_x and A_y are disjoint. For each vertex x of S_{right}, we choose a row R_i such that $i \equiv 1 \pmod 3$, let P_x be the subpath of C_{h+1} from x to $R_i \cap C_{h+1}$ and let $A_x = P_x \cup R_i$. Furthermore, we choose R_i closest to x, i.e. so that there is no $j = 3k+1$ such that $R_j \cap C_{h+1}$ is an interior vertex of P_x. Note that this ensures that if x and y are both in S_{right} then A_x and A_y are disjoint.

Now, if one of $\{a_i, b_i\}$ is in $S_{top} \cup S_{bottom}$ and the other is in $S_{right} \cup S_{left}$ then A_{a_i} and A_{b_i} intersect and we shall choose P_i in their union. If a_i and b_i are both in $S_{top} \cup S_{bottom}$ then we let P_i be a path in $A_{a_i} \cup A_{b_i} \cup R_{3i+2}$. If a_i and b_i are both in $S_{left} \cup S_{right}$ then we let P_i be a path in $A_{a_i} \cap A_{b_i} \cup C_{3i+2}$. It is easy to see that no vertex is in more than two of the P_i. ∎

(5.3). *Let W' be a proper subwall of height h of a wall W such that for any x in W' and any y in $per(W)$, $d_W(x,y)$ is at least $8h$. Let P be the set of pegs of W'. Then for any row of W there are $|P|$ vertex disjoint paths between P and this row in $W - (W' - P)$.*

Proof. The proof is simple and we omit the details. We remark only that each of the paths will be contained either in a column, a column and a row, or two columns and a row.

Robertson and Seymour(see [10], Theorem 9.6) proved:

Theorem 5.4. *For every pair of integers l and t there exists an integer $w(l,t) > max(l,t)$ such that the following holds. If $\mathcal{B}$ is a s-preference in a graph H and $\mathcal{B}$ controls no clique minor of order t but does control a wall W of height $w(l,t)$ then for some subset Y of less than $\binom{t}{2}$ vertices of H there are 48 disjoint proper subwalls of W: $W_1, \ldots, W_{48}$ all of which have height l, are disjoint from Y and are flat and dividing in $H - Y$.*

Now, we set $t = 8r+1$ and $l = 20(1000r + 2\binom{t}{2})$. We insist that $p(r) \geq 1 + 2p(r-1) + 16(20^{64w(l,t)^5} + 1)^2$. We want to show:

5.5. *There is a wall W in G', a set S of $2r$ vertices of $per(W)$ any pair of which are at internal distance at least 10 and each of which is at internal distance 10 from the corners, as well as a set $X=\{x_1,\dots,x_r\}$ of r vertices of C and a set $\mathcal{Q}$ of $2r$ vertex disjoint paths between S and $image(X)$ in $G'-(W-S)$.*

Theorem 4 follows easily from (5.5) and (5.2). To see this, for each x_i let a_i (resp. b_i) be the endpoint in S of the path of $\mathcal{Q}$ whose other endpoint is x_i^A (resp x_i^B). Then letting $\mathcal{P}$ be a set of paths obtained by applying (5.2), we see that the preimage of the union of the paths in $\mathcal{P}\cup\mathcal{Q}$ yields the desired odd cycles.

Proof of 5.5. The first step in the proof of (5.5) is to recall that by (3.5) there is an s-preference B in G' of order $k=20^{64w(l,t)^5}$ such that for any set X of at most k vertices of G', $B(X)$ contains the image of all but at most $(|X|+1)^2$ vertices of C. Furthermore, by (2.2) there is a wall W of height $w(l,t)$ controlled by this preference. By (4.2) and (5.4), we can find a subset Y of at most $\binom{t}{2}$ vertices of G' and 48 disjoint proper subwalls of W: $W_1,\dots,W_{48}$ each of height l such that each W_i is disjoint from Y and flat and dividing in $G'-Y$.

We note that the compasses of the W_i in $G'-Y$ are disjoint and so we can choose one of them, say W_1 whose compass in $G'-Y$ contains at most a sixth of the vertices of $image(C)$. We note also that it is easy to find a proper subwall W_1' of W_1 of height $(1000r+2\binom{t}{2})=\frac{l}{20}$ such that for every x in $per(W_1')$ and every y in $per(W_1)$, $d_W(x,y)\geq 8(1000r+2\binom{t}{2})$. Now, because W_1 is flat and dividing in $G'-Y$, so is W_1'. Let P be the set of pegs of W_1'. Choose a maximal set A of vertices of $C-preimage(W_1)$ such that there is a set $\mathcal{P}$ of $2|A|$ vertex disjoint paths from P to $image(A)$ in $G'-(W_1'-P)-Y$. We claim that:

5.6. $|A|\geq 2000r$.

Now, (5.5) follows easily from this claim for we can greedily construct a subset A' of $\frac{|A|}{40}-80\geq 50r-80$ elements of A such that for any two paths P_1 and P_2 of $\mathcal{P}$ each with one endpoint in A', the endpoint of P_1 on $per(W_1')$ is at internal distance at least 10 from the endpoint of P_2 on $per(W_1')$, and at internal distance at least 10 from any corner of $per(W_1')$ (recall that $r\geq 2$).

Proof of 5.6. To prove (5.6), we need the following technical result:

5.7. *If Z is a cutset of fewer than $|P|$ vertices separating some set L of vertices of $G'-W_1$ from P in $G'-Y-(W_1'-P)$ then L is disjoint from $B(Z\cup Y)$.*

Proof. By (5.3) there are $|P|$ vertex disjoint paths in $W-(W_1'-P)$ from P to every row of W completely contained in $B(Z\cup Y)$, i.e. every row of W disjoint from Z. Thus if there is a vertex v in $L\cap B(Z\cup Y)$ then there is a path Q from v to a vertex of P in $G'-Z-Y$. Now, W_1 is flat and dividing in $G'-Y$ and v is not in W_1. So there is a subpath of Q with one endpoint in $per(W_1)$, and the other in W_1' whose interior is contained in $comp(W_1)-per(W_1)$. Consider such a subpath Q'. Because W_1 is flat, for any subpath R of Q' with its endpoints on W_1 but otherwise disjoint from W_1, there is a brick of W_1 containing both endpoints of R. Recall that for any $x \in per(W_1)$ and $y \in per(W_1')$, $d_W(x,y) > 8000r$. So, if we follow Q from v, then at some point before it intersects W_1', it must intersect a brick B of W_1 disjoint from Z such that B intersects either a row R or column C of W disjoint from Z. If B intersects a row R disjoint from Z then an initial segment of Q along with a path in $B \cup R$ and a path from R to P in $W - Z - (W_1' - P)$ (which exists by 5.3) yields a path from v to P in $G' - Z - Y - (W_1' - P)$, a contradiction. If B intersects a column C disjoint from Z then a similar contradiction arises, although we will need to consider C and a row R disjoint from Z. ∎

We choose a maximal set X of vertices of $C' = C - preimage(W_1 \cup Y)$ such that there is no set Z of fewer than $2|X|$ vertices with $image(X)$ disjoint from $B(Z \cup Y)$. By (5.7), if X has more than $2000r$ vertices then we are done. We assume the contrary and derive a contradiction. The proof is essentially the same as that of 4.2.

Assume first that X is empty. In this case, let x be any vertex of C' whose image is contained in $B(Y)$ (such a vertex exists by Theorem 3.5 and the fact that at least two thirds of the vertices of C are in C'). We know that there is a vertex y such that $image(x)$ is disjoint from $B(Y+y)$. Let P be a path between x^A and x^B in $B(Y)$. Clearly P is disjoint from $B(Y+y)$. However, this contradicts (3.3). So, we can assume that X is non-empty.

Otherwise, let Z_0 be a set of $2|X|$ vertices such that $image(X)$ is disjoint from $B(Z_0 \cup Y)$ chosen so that $B(Z_0 \cup Y)$ is as small as possible (note that $Z_0 = image(X)$ may occur). Now, $B(Z_0 \cup Y)$ contains the image of all but at most $(|Z_0 \cup Y|+1)^2$ vertices of C. Furthermore, W_1 contains at most one sixth of the elements of $image(C)$. Thus, if we choose $p(r)$ large enough, $B(Z_0 \cup Y)$ must contain the image of at least $\frac{p(r)}{2}$ elements of C'.

For each x in C' whose image is contained in $B(Z_0 \cup Y)$, we know by the maximality of X that there is a set Z_x of at most $2|X|+1$ vertices such that $image(X+x)$ is disjoint from $B(Z_x \cup Y)$. We choose Z_x so as to minimize the size of $B(Z_x \cup Y)$ subject to $image(X+x) \cap B(Z_x \cup Y) = \emptyset$.

We claim that for each x with $image(x) \subseteq B(Z_0 \cup Y) - W_1$, Z_x is contained in $B(Z_0 \cup Y) \cup Z_0$ and $Z_x \cup Y$ also separates Z_0 from $B(Z_X \cup Y)$. To see this, let A_1 be the set of vertices of $Z_0 \cup Z_x$ incident to a vertex of $B(Z_0 \cup Z_x \cup Y)$. Let A_2 be the set of vertices of $Z_0 \cup Z_x$ in $image(X)$ or incident to a component of $G' - Z_0 - Z_x - Y$ containing a vertex of $image(X)$. Note that a vertex which is in both A_1 and A_2 is also in both Z_0 and Z_x. Thus, $|A_1| + |A_2| \leq |Z_0| + |Z_x|$. Now, our choice of X ensures that $|A_2| \geq 2|X|$. Thus, $|A_1| \leq |Z_x|$ and, since $B(A_1 \cup Y) = B(Z_0 \cup Z_x \cup Y) \subseteq B(Z_x \cup Y)$, our choice of Z_x ensures that $Z_x = A_1$. Since Z_0 is disjoint from $B(A_1 \cup Y)$, our claim follows.

We also claim that there is no x with $image(x) \subseteq B(Z_0 \cup Y) - W_1$ such that $Z_0 \subseteq Z_x$. To see this, assume the contrary. Then $Z_x - Z_0$ is a vertex y. Now, there is a path between x^A and x^B in $B(Z_0 \cup Y)$ and obviously any such path is disjoint from $B(Z_x \cup Y)$. But this contradicts (3.3).

If we set $s = (\binom{t}{2} + 4000r + 2)^2$ and insist that $p(r)$ is at least $s(2^{4000r}) \geq s(2^{2|X|})$ then we can find a set of s vertices of C, $X' = \{x'_1, \ldots, x'_s\}$ such that for $1 \leq i \leq s$ we have $image(x'_i) \subseteq B(Z_0 \cup Y) - W_1$ and $Z_{x'_i} \cap Z_0 = Z_{x'_1} \cap Z_0$.

We claim that we can assume there is a j such that $Z_{x'_j} \neq Z_{x'_1}$. Otherwise, by (3.3), the s vertices of X' are all in $Crosses(Z_{x'_1} \cup Y)$. However, this contradicts (3.2) for $Z_{x'_1} \cup Y$. This proves our claim, we can assume $j = 2$. We let $Z_1 = Z_{x'_1}$ and $Z_2 = Z_{x'_2}$.

Now, let Z_3 be the set of vertices of $Z_1 \cup Z_2$ adjacent to a vertex of $B(Z_1 \cup Z_2 \cup Y)$. Let Z_4 be the set of vertices of $Z_1 \cup Z_2$ which are either in Z_0 or incident to a vertex in a component of $G' - Z_1 - Z_2 - Y$ containing a vertex of Z_0. Note that if a vertex is in both Z_3 and Z_4 then it is in both Z_1 and Z_2. Thus $|Z_1| + |Z_2| \geq |Z_3| + |Z_4|$. Now, $B(Z_1 \cup Z_2 \cup Y)$ is contained in $B(Z_1 \cup Y)$ and $B(Z_2 \cup Y)$ and one of these containments is strict because $Z_1 \neq Z_2$. Thus, by the choice of Z_1 and Z_2, we have $|Z_3| \geq 2|X| + 2$. So $|Z_4| = 2|X|$, and by our choice of Z_0, $Z_4 = Z_0$. Recall that $Z_2 \cap Z_0 = Z_1 \cap Z_0$, so we have obtained $Z_0 \subseteq Z_1$. However, we have already shown this to be impossible, so we arrive at a contradiction and hence (5.6) holds. ∎

As we have noted, (5.5) follows. ∎

6. The proof of Theorem 5

We turn now to the proof of Theorem 5. We actually prove the following stronger theorem.

Theorem 6.1. *For all $h \geq 3, l$ there exists an f(l,h) such that if G is a graph of maximum degree eight then G contains either an odd cycle cover with at most $f(l,h)$ elements, l vertex disjoint odd cycles, or an Escher wall of height h.*

Now, Theorem 5 follows from this theorem with $q(s) = 4f(s,s) + 1$. To see this, note that any graph formed from the union of $4f(s,s) + 1$ odd cycles, no five of which have a common intersection, has maximum degree eight and no odd cycle cover with fewer than $f(s,s) + 1$ elements.

We shall prove Theorem 6.1 for each h separately by induction on l, assuming that $f(l-1,h)$ exists and defining the value of $f(l,h)$ as a function of a number of parameters including $f(l-1,h)$. Instead of giving an explicit definition, we simply choose the minimum $f(l,h)$ which satisfies a number of inequalities scattered throughout this section. We note that we can assume $l \geq 2$, as we can set $f(0,h) = 0$ and $f(1,h) = 0$. To ease our exposition, we shorten $f(l,h)$ to $f(l)$ in what follows.

So we assume the theorem is false for the given h and l, and consider a minimal counterexample G, corresponding minimum odd cycle cover C, and partition of $G-C$ into two stable sets A' and B'. We construct an auxiliary graph $G^* = G^*(C,A',B')$ as follows. We enumerate C as $\{x_1,\ldots,x_{|C|}\}$ For each vertex x_i of C we define $n_{x_i} = |(N(x_i)\cap A')\cup\{x_j|x_j \in (C\cap N(x)), j>i\}|$. We define the *image* of x as a set of n_x+1 new vertices consisting of x^A and a set S_x^B of n_x vertices. The vertex set of G^* is $V(G)-C\cup(\cup\{image(x)|x\in C\})$. The edge set of G^* has $|E(G)|$ elements, it consists of $E(G-C)$, and for each $x\in C$, an edge from x^A to each neighbour of x in B', an edge from each neighbour of x in A' to an element of S_x^B, and for each edge x_ix_j with $i<j$, an edge from x_j^A to an element of $S_{x_i}^B$. We ensure that every vertex of S_x^B is incident to exactly one edge of G^*. We let $A=A'\cup\{x^A|x\in C\}$ and let $B=B'\cup(\cup\{S_x^B|x\in C\})$. Then, (A,B) is a bipartition of G^*.

Remark. *By splitting the vertices of C into up to eight vertices instead of just two, we ensure that for any set $\mathcal{P}$ of disjoint paths in G^*,there is no vertex x of C such that two elements of $\mathcal{P}$ contain elements of $image(x)$ on their interiors. This allows us to mimic the proof of Theorem 4 to find the desired vertex disjoint odd cycles or Escher wall.*

The preimages and images of sets of vertices and subgraphs are defined analogously to the definitions for G'. For $X\subseteq V(G^*)$, we say a vertex x in C *crosses* X if there is no path P of G^*-X with both endpoints in $image(x)$ whose preimage is an odd cycle in G, i.e. if there is no component of G^*-X containing both x^A and a vertex of S_x^B. We let $Crosses(X)$ be the set of vertices of C which cross X. We say $x\in C$ *crosses a separation* (W,Y) if neither $V(W)-V(Y)$ nor $V(Y)-V(W)$ contains both x^A and an element of S_x^B.

We let $Cross(W,Y)$ be the set of vertices of C which cross (W,Y).

Following the proof of (3.2), we can show:

6.2. *For any $X\subseteq V(G^*)$, the set $Crosses(X)$ has at most $7|X|(|X|+1)$ elements.*

The extra factor of seven here comes from the fact that the image of a vertex of C may have as many as eight elements.

Now, mimicing the proof of (3.5), we obtain:

Theorem 6.3. *There is a s-preference B assigning to each subset X of at most $\frac{\sqrt{f(l)-2f(l-1)}}{7}-1$ vertices of G', the component of $G-X$ which contains both x^A and an element of S_x^B for all but at most $7(|X|+1)^2$ of the x in C.*

Next, mimicing the proof of (4.2), we obtain:

6.4. *B controls no clique minor of order $8l+1$.*

Remark. *In mimicking (4.2), we will consider a maximal set X such that, in the graph G^*_X obtained from G^* by adding, for each vertex $x \in X$ a vertex x_b adjacent to all the vertices of S^B_x, there is no cutset of fewer than $2|X|$ vertices separating $X' = \cup\{x^A, x^B | x \in X\}$ from the clique minor. The rest of the proof proceeds in G^*_X. Having defined Z_0 and U_0, we know that for each x with $x^A \in U_0$ and $S^B_x \cap U_0 \neq \emptyset$, there is a set Z_x of $2l+1$ vertices of G^*_{X+x} separating $X' + x^A + x^B$ from the clique minor. We sketch below a proof that $x^B \notin Z_x$ and hence that Z_x is actually a set of vertices separating $X' \cup image(x)$ from the clique minor in G^*_X. This permits us to mimic the proof of (4.2), as it shows our new method for choosing X makes no difference.*

To see that x_B is not in Z_x, we note first that $Z_x \cap U_0$ is non-empty as $x_A \in U_0$. If x_B were in Z_x, then $Z_x - x_B$ would be a cutset of size $|X'|$ separating X' from the clique minor. But this yields a contradiction using once again the argument we have applied so often, whose details we omit.

Next, for any k, if we set $t = 12l + 3$ and $m = 200(1000k + 2\binom{t}{2})$ then we can mimic the proof of (5.5) to prove:

6.5. *If $f(l) \geq 2f(l-1) + 28(20^{64w(m,t)^5} + 1)^2$ then there exists a flat wall W in G^*, a set S of $2k$ vertices of $per(W)$ any pair of which are at interior distance 10 and each of which is at interior distance at least 10 from the corners of W, as well as a set $X = \{x_1, \ldots, x_k\}$ of vertices of C and a set $\mathcal{Q}$ of $2k$ vertex disjoint paths of G^* between S and $image(X)$ such that for $1 \leq i \leq k$ there is a path in $\mathcal{Q}$ with endpoint x^A_i and a path in $\mathcal{Q}$ with an endpoint in $S^B_{x_i}$.*

Remark. *In the proof of 6.5, we need the 48 disjoint walls guaranteed by 5.4 because if a wall contains $\epsilon|image(C)|$ elements of $image(C)$ then it may intersect the image of up to $8\epsilon|C|$ elements of C.*

Setting $k = l^2$ and ensuring that the appropriate bound on $f(l)$ holds, we obtain:

6.6. *There exists a flat wall W in G^*, a set S of $2l^2$ vertices of $per(W)$ any pair of which are at interior distance 10 and each of which is at interior distance at least 10 from the corners of W, as well as a set $X = \{x_1, \ldots, x_{l^2}\}$ of vertices of C and a set $\mathcal{Q}$ of $2l^2$ vertex disjoint paths of G^* between S and $image(X)$ in $G - (W - S)$ such that for $1 \leq i \leq l^2$ there is a path in $\mathcal{Q}$ with endpoint x^A_i and a unique path in $\mathcal{Q}$ with an endpoint in $S^B_{x_i}$.*

Now, for each x_i in X, we let a_i be the endpoint on $per(W)$ of the path of $\mathcal{Q}$ with one endpoint x^A. We let b_i be the endpoint on $per(W)$ of the path of $\mathcal{Q}$ with one endpoint in S^B_x. We say (a_i, b_i) crosses (a_j, b_j) if these four vertices appear around $per(W)$ in the cyclic order (a_i, a_j, b_i, b_j). We can construct a partial order on the (a_i, b_i) pairs such that two pairs are incomparable if and only if they cross. It follows from Dilworths Theorem [2] that either there is a set of l pairs no two of which cross or there is a set of l pairs every two of which cross. We consider these two cases separately.

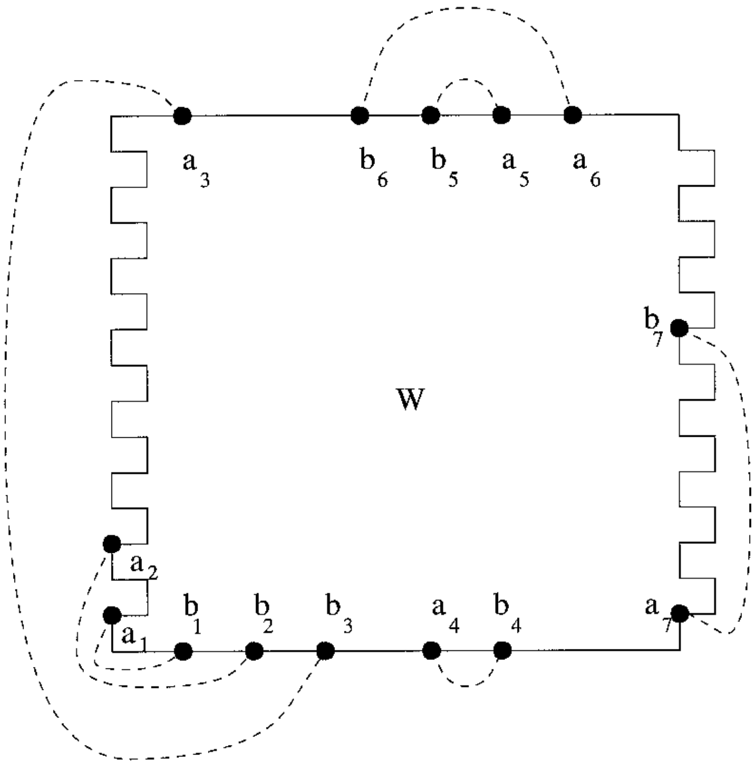


Figure 5

Case 1. *There is a set of l non-crossing pairs.*

We can assume these are the pairs $(a_1, b_1), \ldots, (a_l, b_l)$, see Fig. 5 for an example. Now, simple brute force techniques along the lines of the proof of (5.2) (or more powerful machinery, see e.g. [8]) can be used to show that there are l vertex disjoint paths $P_1, \ldots, P_l$ of W such that P_i links a_i and b_i. But now concatenating these paths with the paths in Q, we obtain a set $Q_1, \ldots, Q_l$ of disjoint paths such that Q_i links x^A to an element of S_x^B. The preimages of these paths yield l vertex disjoint odd cycles in G. This contradiction shows that this case cannot occur.

Case 2. *There is a set of l crossing pairs.*

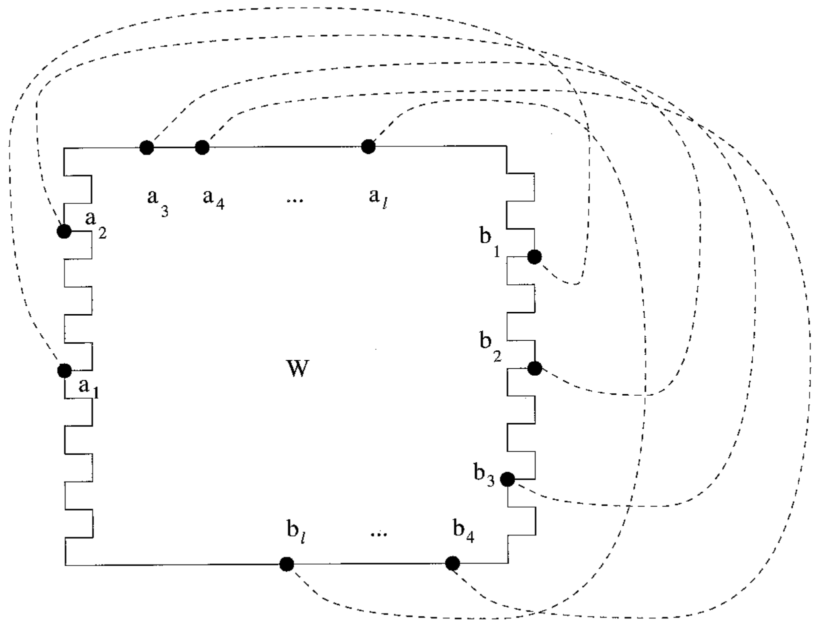


Figure 6

We can assume these are the pairs $(a_1,b_1),\ldots,(a_l,b_l)$. We note that the preimage of W is a bipartite wall of G, of height at least $4l^2$ (because of 6.6). Thus the preimage of $W \cup (\cup\{Q|Q \in \mathcal{Q}\})$ is a bipartite wall together with a set $\mathcal{P}$ of l paths each of which joins two points a_i and b_i on the perimeter of the wall and such that adding any one of these path to the wall creates an odd cycle. Now, a_i and b_i are symmetric and because every pair crosses we can relabel the endpoints of each of the l paths so that these endpoints appear around $per(W)$ in the order $a_1,a_2,\ldots,a_l,b_1,b_2,\ldots,b_l$ (see Fig. 6 for an example). Now, let W' be a proper subwall of W of height l chosen so that no vertex of W' is at distance less than $4l$ from the perimeter of W. Let E be the set of pegs on the top row of W' and let F be the set of pegs on the bottom row of W'. Now, tedious brute force techniques (or more powerful machinery, see e.g. [8]) can be used to show that there are $2l$ vertex disjoint paths $P_1,\ldots,P_{2l}$ of $W-(W'-E-F)$ such that for $1 \le i \le l$, P_i links a_i and a point of E whilst for $l+1 \le i \le 2l$, P_i links b_i and a point of F. Now the planarity of W and our condition on the parity of the paths in $\mathcal{P}$ ensures that the preimage of $W' \cup (\cup\{P|P \in \mathcal{P}\}) \cup (\cup_{i=1}^{2l} P_i)$ is an Escher wall of height l. Thus, this case cannot occur. This completes the proof of Theorem 5 and hence Theorem 1.

7. The proof of Theorem 3

Consider any Escher wall consisting of a wall W and paths $P_1,\ldots,P_h$. We can easily prove, along the lines of (5.2), that there are h paths $Q_1,\ldots,Q_{h-1}$ in W such that P_i and Q_i have the same endpoints and no vertex of W is in more than two of the Q_i. It follows that this Escher wall contains a half-integral packing of $\lfloor\frac{h-1}{2}\rfloor$ odd cycles. Now Theorem 3 follows immediately from Theorem 1 with $n_k = t(k, 2k+1)$.

For those who wish a formal proof consider:

For each i between 1 and $h-1$, let A_i consist of the subpath of R_1 between the endpoint a_i of P_i on R_1 and the ith nail of R_1 concatenated with a subpath of C_i between this nail and a nail $n_{i,i+1}$ which is in the intersection of C_i and R_{i+1}. Let B_i consist of the subpath of R_{h+1} between the endpoint b_i of P_i on R_{h+1} and the $h+2-i$th nail of R_{h+1} concatenated with a subpath of C_{h+2-i} between this nail and a nail $n_{h+2-i,i+1}$ which is in the intersection of C_{h+2-i} and R_{i+1}. Let D_i be the subpath of R_{i+1} between $n_{i,i+1}$ and $n_{h+2-i,i+1}$. Let Q_i be a path between a_i and b_i in $A_i \cup B_i \cup D_i$. Then $P_i \cup Q_i$ is an odd cycle. We claim that no vertex is in more than two of these odd cycles. Clearly, it is enough to show that no vertex is in more than two Q_i. So, consider a vertex v. Now, v is in at most one A_i, one B_j and one D_k. Furthermore, if v is in $A_i \cap B_j$ then $i > \frac{h}{2}+1$ and $j = h+2-i$. Also, v is either in no R_l or in $C_i \cap R_l$ for some l between $h+3-i$ and $i+1$. Thus, if v is in D_k for $k \notin \{i,j\}$ then $k \in \{h+3-i,\ldots,i-1\}$. But no such D_k intersects C_i as D_k runs between C_k and C_{h+2-k}. So v is in at most two of the odd cycles, as claimed. ∎

8. The proof of Theorem 2

In this section, we show that, for every k, sufficently high Escher walls are not k-near bipartite. To do so, we need to introduce some technical results.

Definition. We say G is an *odd subdivision* of H if G is a subdivision of H such that the paths of G corresponding to the edges of H all have an odd number of edges.

8.1. *If G is an odd subdivision of H then G is k-near bipartite if and only if H is.*

Proof. To prove this, we need only consider G and H where G is obtained from H by subdividing the edge ad to obtain the path $abcd$. The general result then follows by recursively applying this restricted version.

We show first that if G is k-near bipartite then so is H. To this end consider any subgraph F of H. If the edge ad is not in F then F is also a subgraph of G and hence contains a stable set with $\frac{|V(F)|-k}{2}$ vertices. If ad is an edge of F then let F' be the subgraph of G obtained from F by deleting the edge ad, and adding the vertices b, c along with the edges ab, bc, cd. By assumption, F' contains a stable set with $\frac{|V(F')|-k}{2}$ vertices. We can obtain such a stable set S which does not contain both a and d (by removing a and adding b if neccesary). Now, $S \cap F$ is stable in F and clearly $|S \cap F| \geq \frac{|V(F)|-k}{2}$. So, H is k-near bipartite.

Now, assume H is k-near bipartite and let F be a subgraph of G. We want to show that F contains a stable set with $\frac{|V(F)|-k}{2}$ vertices. Clearly, we can restrict our attention to graphs without vertices of degree 0 or 1. Thus, if F is not a subgraph of H then the edges ab, bc, and cd are all present in F. Now by considering a maximum stable set in $F - b - c + ad$ and adding one of b or c, we obtain a stable set in F of the desired size.

Definition. A wall W is *basic* if it is obtained from an elementary wall by subdividing every edge once. An Escher wall $(W, P_1, \ldots, P_k)$ is *basic* if W is basic and each P_i is an edge joining two vertices both of which are at distance 2 along $per(W)$ from two nails. Thus, every vertex on one side of the bipartition of W has degree 2 in the Escher wall, and every vertex on the other side of the bipartition is a corner or has degree 3 in the Escher wall.

8.2. *For all h there is a $w(h)$ such that every wall W of height $w(h)$ contains a subwall W^* which is an odd subdivision of the basic wall of height h such that the perimeter of W^* separates its interior from $per(W)$.*

Proof. See [15]. The theorem stated in [15] does not contain the condition on the perimeter of W^* but the proof clearly yields the strengthened statement. ∎

8.3. *For each l there exists an $f(l)$ such that an Escher wall W of height at least $f(l)$ contains an odd subdivision of a basic Escher wall of height l.*

Proof. Consider an Escher wall $(W, P_1, \ldots, P_{4w(20h^2)})$ of height $4w(20h^2)$. We can find within W, a proper subwall W' of height $w(20h^2)$ which is at internal distance $w(20h^2)$ from the perimeter of W. By (8.2), we can find a subwall W^* of W' of height $20h^2$ which is an odd subdivision of a basic wall and whose perimeter separates its interior from $per(W')$ and hence also $per(W)$. We can take every $(10h)th$ column of W^* and every $(10h)$th row starting with R_{20h} to obtain an odd subdivision W_1 of a basic wall of height h such that the path between every two nails in the top and bottom rows of W_1 contains a special nail of W^* such that (i) each special nail is incident to a vertex in $W^* - W_1$, (ii) by our condition on W^*, the special nails are on the same side of the bipartition as the nails of W_1. and (iii) every two special nails are separated by a nested set of $10h$ circuits of W^*.

It is now easy to prove, by considering the subgraph of W embedded in the cylinder bounded by $per(W)$ and $per(W_1)$ that there is a set $\mathcal{Q}$ of $2h$ paths between the endpoints of these special nails and the endpoints of $P_1, \ldots, P_h$ in W, so that the endpoint of P_i on the top (resp. bottom) row of W is linked to the ith special nail on the top row of W_1 (resp. $(h+1-i)th$) special nail on the bottom row of W_1). We omit the details, and just sketch the idea of the proof. First, Menger's theorem allows us to prove that there are $2h$ vertex disjoint paths in this subgraph linking the special nails to the endpoints of the P_i. To see this, it is important to recall that we strengthened (8.2) to ensure that this cylinder is disjoint from W_1. It is then easy to show that because our special nails are separated by nested sets of $10h$ disjoint circuits of W^*, a cutset of size less than $2h$ which shows the desired paths do not exist would also separate all of W_1 from per(W) and W_1 simply has too many nails for this to happen.

Now, the planarity of W ensures that the paths we have found would match the desired endpoints provided we simply applied a cyclic shift to one of the two sets of endpoints. We can do this by twisting the paths in $W - W'$ (using brute force techniques, or the more sophisticated techniques of [8]). It follows that $W_1 \cup \{Q|Q \in \mathcal{Q}\} \cup \{P_1, \ldots, P_h\}$ yields an Escher wall. Our choice of W_1 and the endpoints of the elements of $\mathcal{Q}$ ensure that it is an odd subdivision of a basic Escher wall. ∎

By virtue of (8.1) and (8.3), to prove Theorem 2, we need only prove:

8.4. *For every k there is an $h(k)$ such that a basic Escher wall of height $h(k)$ is not k-near bipartite.*

Proof. We consider a graph F_l such that:

(i) Its vertex set can be partioned into $2l^2+2l$ cycles $C_1, \ldots, C_{2l^2+2l}$ each containing $8l^2+8l-2$ vertices, and $2l+1$ trees $T_1, \ldots, T_{2l+1}$ each containing $6l-5$ vertices,

(ii) The cycle C_i is obtained by concatenating two paths $P_{i,1}$ and $P_{i,2}$ each containing $4l^2+4l-1$ vertices (that is: the last vertex of $P_{i,3-j}$ is adjacent to the first vertex of $P_{i,j}$ for $j \in \{1,2\}$). We use $v_{i,j,k}$ to denote the kth vertex of $P_{i,j}$.

(iii) Consider i and j with $1 \le i \le 2l^2+2l-1$, $j \in \{1,2\}$ If i is odd then for $1 \le r \le 2l^2+2l-1$, the $2r^{th}$ vertex of $P_{i,j}$ is adjacent to the $2r^{th}$ vertex of $P_{i+1,j}$. If i is even then for $1 \le r \le 2l^2+2l$, the $(2r-1)^{st}$ vertex of $P_{i,j}$ is adjacent to the $(2r-1)^{st}$ vertex of $P_{i+1,j}$.

(iv) For $1 \le j \le 2l+1$, we define the sets $S_1^j = \{v_{1,1,2jl+2t-1} | 1 \le t \le l\}$, and $S_2^j = \{v_{1,2,2jl+2t-1} | 1 \le t \le l\}$.

(v) For each j between 1 and $2l+1$, $S_1^j \cup S_2^j \cup V(T_j)$ induces a tree T'_j. The leaves of T'_j are the elements of $S_1^j \cup S_2^j$. Furthermore, T'_j is obtained from a tree T''_j all of whose internal nodes have degree 3 by subdividing each edge once.

We shall show:

8.5. *There is an $f(l)$ such that every basic Escher wall of height $f(l)$ contains an odd subdivision of F_l.*

and:

8.6. *F_l is not $(l-1)$-near bipartite.*

Combining these results yields (8.4). We prove (8.6) first.

Proof of 8.6. For each j between 1 and $2l+1$, we let X_j be the set of vertices of T'_j of degree 2, i.e. those obtained in subdividing the edges of T''_j. It is easy to see that we can find a maximum stable set S in F_l such that for all j with $1 \le j \le 2l+1$, either $X_j \cap S = \emptyset$ or $X_j \cap S = X_j$. If $X_j \cap S = \emptyset$ then $S \cap V(T_j) = V(T_j) - X_j$. Now, $|X_j| = \frac{|V(T'_j)|}{2} - \frac{1}{2} = \frac{|V(T_j)|}{2} - \frac{1}{2} + l$. So, if we let n_1 be the size of $\{j | S \cap V(T_j) = X_j\}$ and n_2 be the size of $\{j | S \cap V(T_j) = V(T_j) - X_j\}$ then $|S \cap (\cup_{j=1}^{2l+1} V(T_j))| = \frac{1}{2}|\cup_{j=1}^{2l+1} V(T_j))| + n_1(l-\frac{1}{2}) - n_2(l-\frac{1}{2})$. We shall prove:

8.7. $|S \cap (\cup_{j=1}^{2l^2+2l} V(C_j))| \le \frac{1}{2}|\cup_{j=1}^{2l^2+2l} V(C_j)| - ln_1$.

It follows that $|S| \le \frac{1}{2}|V(F_l)| - \frac{n_1}{2} - (l-\frac{1}{2})n_2 \le \frac{1}{2}|V(F_l)| - l - \frac{1}{2}$. Hence, we see that F_l is not $(l-1)$-near bipartite.

To prove 8.7, we consider each C_i separately. We can assume $n_1 \ge 1$ as otherwise, since each C_i has a perfect matching, we have: $|S \cap (\cup_{j=1}^{2l^2+2l} V(C_j))| \le \frac{1}{2}|\cup_{j=1}^{2l^2+2l} V(C_j)|$ and we are done.

Note first that the only maximum stable sets of C_i consist of the odd indexed vertices from one of the $P_{i,j}$ and the even indexed vertices from the other. Thus if there is a pair of vertices, one from each of the $P_{i,j}$, missed by S and with the same "parity" both of which are not in S then $|S \cap C_i| \le \frac{|V(C)|}{2} - 1$.

More generally, we have the following result whose easy proof we omit:

8.8. *Consider a set A_i of $r_i > 0$ vertices of $P_{i,1} - S$ and a set B_i of $s_i > 0$ vertices of $P_{i,2} - S$ all of whose indices have the same parity. Enumerate A_1 as $\{a_1, \ldots, a_{r_i}\}$ and B_i as $\{b_1, \ldots, b_{s_i}\}$, so that the vertices of A_i (resp. B_i) appear in the given order along $P_{i,1}$ (resp. $P_{i,2}$). Let X be a set of vertices of C_i disjoint from S such that:*

(i) *the parity of the indices of the vertices of X differs from that of those in $A_i \cup B_i$,*
(ii) *No two elements of X lie in the same component of $C_i - A_i - B_i$,*
(iii) *the components of $C_i - A_i - B_i$ bounded by $\{a_1, b_{s_i}\}$ and $\{b_1, a_{r_i}\}$ are disjoint from X.*

Then $|S \cap C_i| \le \frac{1}{2}|C_i| - |X| - 1$.

Now, we let $A_1 = \cup\{S_1^j | 1 \le j \le 2l+1, S \cap V(T_j) = X_j\}$. Similarly, we let $B_1 = \cup\{S_2^j | 1 \le j \le 2l+1, S \cap V(T_j) = X_j\}$. We can choose a set Y_1 of $|A_1| - 1$ even indexed vertices of $P_{1,1}$ such that every component of $C_1 - A_1$, except that containing $P_{1,2}$ contains exactly one vertex of Y_1. We can choose a set Z_1 of $|B_1|-1$ even indexed vertices of $P_{1,2}$ such that every component of $C_1 - B_1$, except that containing $P_{1,1}$ contains exactly one vertex of Z_1. We let X_1 be the set of vertices of $Y_1 \cup Z_1$ which are not in S. By (8.8), we have $|S \cap C_1| \leq \frac{1}{2}|V(C_1)| - |X_1| - 1$. Now, we let A_2 be the set of neighbours of $Y_1 - X_1$ on C_2 and let B_2 be the set of neighbours of $Z_1 - X_1$ on C_2. We can define Y_2, Z_2, and X_2 analagously to Y_1, Z_1, and X_1 (they will of course have odd indices). We can recursively define A_k, B_k, Y_k, Z_k, X_k such that $|A_k| = |A_{k-1}| - |X_{k-1} \cap Y_{k-1}| - 1$, $|B_k| = |B_{k-1}| - |X_{k-1} \cap Z_{k-1}| - 1$, and $|S \cap C_i| \leq \frac{1}{2}|V(C_i)| - |X_i| - 1$ if A_i and B_i are non-empty. Letting t be the largest index such that A_t and B_t are both non-empty and assuming without loss of generality that $A_{t+1} = \emptyset$, we see that

$$|S \cap (\cup_{j=1}^{2l^2+2l} V(C_j))| \leq \frac{1}{2}|\cup_{j=1}^{2l^2+2l} V(C_j)| - \sum_1^t (|X_i| + 1)$$

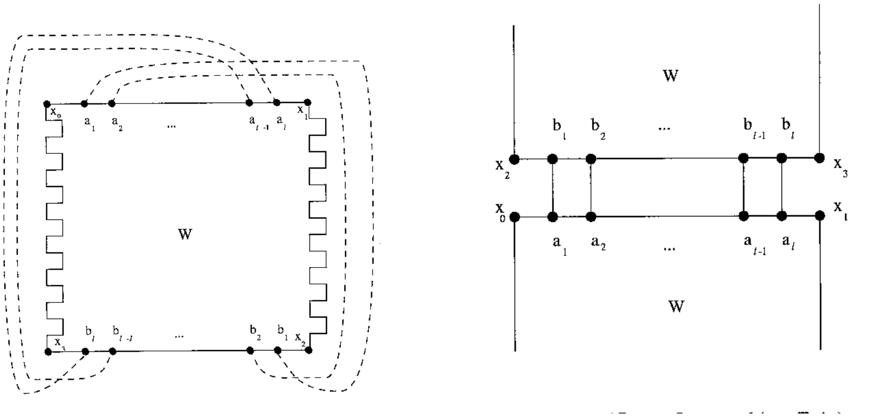
and:

$$\sum_1^t (|X_i| + 1) \geq \sum_1^t (|A_i| - |A_{i+1}|) = |A_1| = ln_1.$$

Thus, (8.7) does indeed hold. ∎

Hence so does (8.6). ∎

Proof of 8.5. To formally present this result would be a tedious business. We shall simply present the intuition behind the proof, mainly through the use of diagrams.



(a) The wall

(b) Top and Bottom Juxtaposed (note Twist)

Figure 7.

We want to show that any sufficently large elementary Escher wall contains an odd subdivision of an F_l. The C_i will be formed using the top and bottom portions of the wall as well as the edges between the top and bottom rows. The T_j will run through the middle of the wall. We pictorially depict how to choose the C_i. To do so, we consider cutting the wall through the middle and a diagram which juxtaposes the top and the bottom of the wall, see Fig. 7.

Now, we choose C_1 to run through the Escher wall as depicted in Fig. 8. We can layer two cycles as depicted in Fig. 9. More generally, if the wall is large enough, we can layer as many cycles as we want(ensuring C_1 is the cycle closest to the middle of the wall). Furthermore, by choosing paths between C_i and C_{i+1} which lie alternately on the top and bottom of the wall, we can obtain paths which leave and arrive at vertices whose indices alternate in parity (recall that W is basic). Hence, we can find paths between the C_i required to construct the odd subdivision of an F_l.

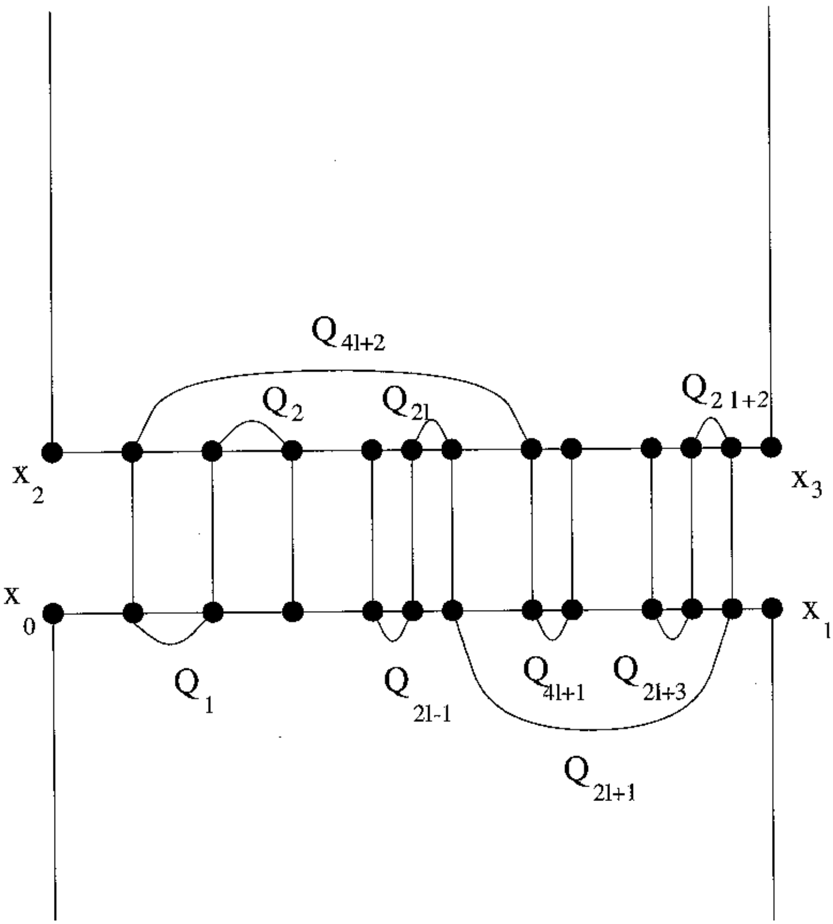


Figure 8. C_1 (recall twists)

Tedious but routine arguments allow us to add the trees in the middle of the wall as suggested in Fig. 10. We omit the details of this proof, the crucial idea is the layering pattern depicted in Figs. 8 and 9. ∎

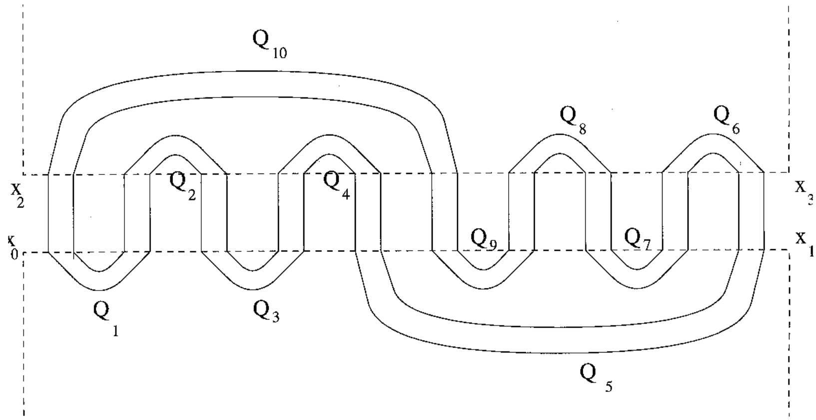


Figure 9. Layering the cycles (recall twist)

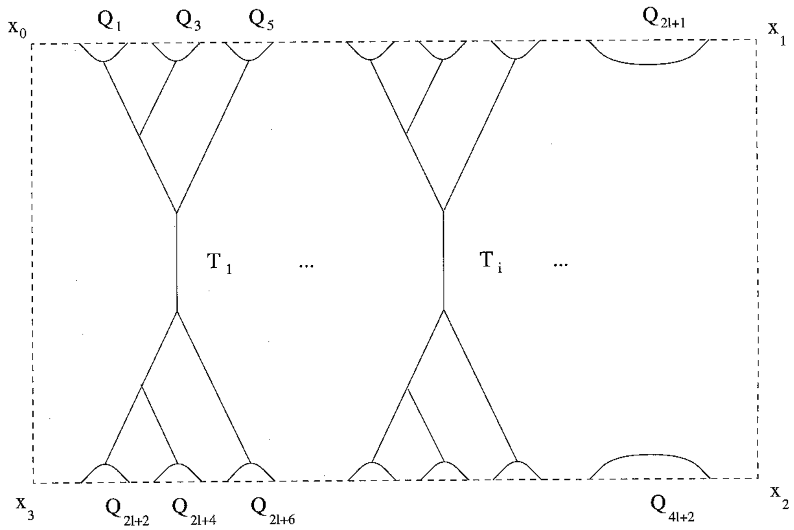


Figure 10. The T_i (no twist)

9. Concluding remarks

We consider first the algorithmic implications of our result. We begin by remarking that determining, given a graph G and an integer k, if G has k vertex disjoint odd cycles is NP-complete. To see this, we note that if $\{s_1,\ldots,s_k,t_1,\ldots,t_k\}$ are $2k$ vertices of a graph H, then the graph obtained from H by subdividing each edge and adding k new edges $e_1,\ldots,e_k$ s.t. e_i has endpoints s_i and t_i, has k vertex disjoint odd cycles if and only if H has k vertex disjoint paths $P_1,\ldots,P_k$ such that P_i has endpoints s_i and t_i. Since the problem of determining if such paths exist (which we call k-Disjoint Rooted Paths) is NP-complete (see [10] for a reference), so is determining if a graph has k vertex disjoint odd cycles. As far as we are aware, it is an open problem to determine if a graph has a half-integral packing of k odd cycles. It is our guess, and rather recklessly our conjecture, that this problem is also NP-complete.

Robertson and Seymour have shown that k-DRP can be solved in polynomial time for fixed k. This, combined with our results, yields a polynomial time algorithm for determining, for k fixed, if G has a half integral packing of k odd cycles. It proceeds as follows.

First, we determine if there is an odd cycle cover with at most n_k vertices for the n_k of Theorem 3. If no such cover exists then we are done, G has a half-integral packing of k odd cycles. Otherwise, we let X be such a cover. We obtain a new graph G' by duplicating every vertex of G (that is replacing each vertex x by two vertices x and x' s.t. $N_{G'}(x)=N_{G'}(x')=\cup\{y,y'|y\in N_G(x)\}$. Clearly, G has a half-integral packing of k odd cycles if and only if G' contains $2k$ vertex disjoint odd cycles. Furthermore, $X'=\{x,x'|x\in X\}$ is an odd cycle cover of G' with at most $2n_k$ vertices. Thus, every odd cycle in G' corresponds to a set of internally vertex disjoint paths between the elements of X'. So to determine if G' has $2k$ vertex disjoint odd cycles, we need only determine for every set of pairs of vertices of X' with each vertex in at most two pairs, whether or not G' contains a set of internally vertex disjoint paths whose endpoints are given by the specified pairing. This can be done in polynomial time (see [10]) and our result follows.

We also remark that Theorem 1 ensures that the Erdős–Pósa property holds for the odd cycles in planar graphs. Thus, the above technique can be used to determine if a planar graph contains k odd cycles for k fixed. As of the current writing, the author and P. Seymour believe they have a much more complicated algorithm for determining if a graph contains k vertex disjoint odd cycles, k fixed. However, the proof of this result is extremely complicated and may well never be written down.

We do not know the complexity of determining if a graph G is k-near bipartite. Again, if k is part of the input, we imagine the problem is NP-complete. Do our results help in solving the problem for fixed k?

Finally, it would be of interest to determine the relationship between k and $f(k)$. As of this writing, the author and Mike Molloy believe they can prove $f(0)=5$ thereby proving a conjecture of Gyárfas. Improving the bounds on $f(k)$ for general

k may be more difficult.Our results give a super exponential bound which is almost certainly far from the truth. The analagous bound for the family of all cycles is $O(k\sqrt{log\ k})$. This may well be the answer for odd cycles too. A similar and perhaps more interesting question is to determine the relationship between the maximal fractional packing of odd cycles and the maximum half-integral packing.

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